REMARKS ON GENERALIZED HARDY ALGEBRAS

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ABSTRACT. For a measure space (Ω, Σ, µ) with a positive finite measure µ, and a positive real number p, we define the space $L^+_p(µ) = L^+_p$ of all (equivalence classes of) Σ-measurable complex functions $f$ defined on Ω such that the function $(\log^+ |f|)^p$ is integrable with respect to $µ$. We define the metric $d_p$ on $L^+_p$ which generalizes the metric introduced by Gamelin and Lumer in [G] for the case $p = 1$. It is shown that the space $L^+_p$ is a topological algebra. On the other hand, one can define on the space $L^+_p$ an equivalent F-norm $|·|_p$ that makes $L^+_p$ into an Orlicz space. For the case of the normalized Lebesgue’s measure $dt/2\pi$ on $[0, 2\pi)$, it follows that the class $N^p(1 < p < \infty)$ introduced by I. I. Privalov in [P], may be considered as a generalization of the Smirnov class $N^p$. Furthermore, $N^p(1 < p < \infty)$ with the associated modular becomes an Hardy-Orlicz class. Finally, for a strictly positive and measurable on $[0, 2\pi)$ function $w$, we define the generalized Orlicz space $L^w_p(dt/2\pi) = L^w_p$ with the modular $\rho^w_p$ given by the function $\psi_w(t, u) = (\log(1 + uw(t)))^p$, with a ”weight” $w$. We observe that the space $L^w_p$ is a generalized Orlicz space with respect to the modular $\rho^w_p$. We examine and compare different topologies induced on $L^w_p$ by corresponding ”weights” $w$.

1. INTRODUCTION

For a measure space (Ω, Σ, µ) with a nonnegative finite, complete measure µ not vanishing identically, denote by $L^p(µ) = L^P$ (0 < p ≤ ∞) the familiar Lebesgue spaces on Ω. In Section 2, for $p > 0$, we define the class $L^+_p(µ) = L^+_p$ of all (equivalence classes of) Σ-measurable complex functions $f$ defined on Ω such that the function $\log^+ |f|$ is in $L^P$. Every space $L^+_p$ is an algebra. For each $p > 0$, in Section 2 we define the
metric $d_p$ on $L^+_p$ by
\[ d_p(f, g) = \inf_{t > 0} \left[ t + \mu \left( \{ x \in \Omega : |f(x) - g(x)| \geq t \} \right) \right] \]
\[ + \int_{\Omega} \left| \left( \log^+ |f(x)| \right)^p - \left( \log^+ |g(x)| \right)^p \right| \, d\mu. \]

The space $L^+_1$ was introduced in [G, p. 122], with the notation $L(\mu)$ in [G]. It was proved in [G] that $L^+_1$ with the topology given by the metric $d_1$ is a topological algebra. In Section 2 we prove the same statement for every space $L^+_p, p > 0$. We also define two metrics $\rho_p$ and $\delta_p$ on $L^+_p$, and we show that they induce the same topology on $L^+_p$ as the initial metric $d_p$.

By analogy with the Hardy algebra $H(\mu)$ defined in [G], in Section 3 we define the algebra $H_p(\mu)$ with $p > 0$. It is known (see [G]) that the Smirnov class $\mathcal{N}^+$ on the unit disk $D : |z| < 1$ in the complex plane may be considered as the Hardy algebra $H(d\theta/2\pi)$. The analogous results are obtained in Section 5 for the algebra $N^p, p > 1$, introduced by I. I. Privalov with the notation $A_q$ in [P].

In Section 4 we note that the function $\psi : [0, \infty) \mapsto [0, \infty)$ defined as $\psi(t) = (\log(1 + t))^p$, is an Orlicz function or a $\varphi$-function. Further, we observe that the space $L^+_p(d\theta/2\pi) = L_p, p > 0$, consisting of all complex-valued functions $f$, defined and measurable on $[0, 2\pi)$ for which
\[
(1.1) \quad (\|f\|_p)^p := \int_0^{2\pi} (\log(1 + |f(t)|))^p \, \frac{dt}{2\pi} < +\infty.
\]
is the Orlicz class coinciding with the associated Orlicz space (see [Mu, Definition 1.4, p. 2]), whose generalization we give in Section 6. We prove that the modular convergence $\| \cdot \|_p$ and the norm convergence $\| \cdot \|_p$ are equivalent. As an application, we show that there does not exist a nontrivial continuous linear functional on the space $(L_p, \| \cdot \|_p)$.

For $p > 1$, following I. I. Privalov (see [P, p. 93]), a function $f$ holomorphic in $D$, belongs to the class $N^p$, if there holds
\[
\sup_{0 \leq r < 1} \int_0^{2\pi} (\log^+ |f(re^{i\theta})|)^p \, \frac{d\theta}{2\pi} < \infty.
\]

In Section 5 we note that the algebra $N^p$ may be considered as the Hardy-Orlicz space with the Orlicz function $\psi$ defined in Section 4. Identifying a function $f \in N^p$ with its boundary function $f^*$, the space $N^p$ is identical with the closure of the space of all functions holomorphic
in $D$ and continuous in $\bar{D} : |z| \leq 1$ in the space $(L_p dt/2\pi) \cap N, |.|_p$). On the other hand, $(N^p, d_p)$ coincides with the space $H_p(dt/2\pi)$ defined in Section 3, i.e. with the closure of the disk algebra $P(\bar{D})$ in the space $(L_p(dt/2\pi), d_p)$. Therefore, the space $N^p$ may be considered as generalized Hardy algebra. From this fact, it is easy to show that $N^p$ is an $F$-algebra with respect to the $F$-norm $\| \cdot \|_p$ given by 1.3.

In the last section we note that the real function $\psi_w$ with $p > 0$, defined on $\Omega \times [0, \infty)$ by the formula

$$\psi_w(t, u) = (\log(1 + uw(t)))^p,$$

is a Musielak-Orlicz function. For the Lebesgue measure space $(\Omega, [0, 2\pi], dt/2\pi)$, we denote by $L^w_p(dt/2\pi) = L^w_p$, $p > 0$, the class of all (equivalence classes of) complex-valued functions $f$, defined and measurable on $[0, 2\pi]$ for which

$$(\|f\|^w_p)^p := \int_0^{2\pi} (\log(1 + |f(t)|w(t)))^p \frac{dt}{2\pi} < +\infty.$$

Then $L^w_p$ is the generalized Orlicz class with the modular $\| \cdot \|^w_p$, and $L^w_p$ coincides with the associated generalized Orlicz space. $\| \cdot \|^w_p$ is a modular in the sense of Definition 1.1 in [Mu, p. 1], and by $\rho^w_p(f, g) = ((\|f - g\|^w_p)^\min\{p, 1\}$, $f, g \in L^w_p$, is defined an invariant metric on $L^w_p$. By [Mu, p. 2, Theorem 1.5, and p. 35, Theorem 7.7], it follows that the functional $| \cdot |^w_p$ defined for $f \in L^w_p$ as

$$|f|^w_p = \inf \left\{ \varepsilon > 0 : \int_0^{2\pi} \left( \log \left( 1 + \frac{|f(t)|w(t)}{\varepsilon} \right) \right)^p \frac{dt}{2\pi} \leq \varepsilon \right\},$$

is a complete $F$-norm. We prove that $(L^w_p, \rho^w_p)$ is an $F$-space for any weight $w$.

For two weights $w$ and $\omega$ such that $\log^+(w/\omega) \in L^p$, we show that $L^w_p \subset L^\omega_p$, and $L^\omega_p = L^w_p$ if and only if $\log(w/\omega) \in L^p$. Further, we prove that the topology defined on $L^\omega_p$ by the metric $\rho^\omega_p$ is stronger than that induced on $L^\omega_p$ by the metric $\rho^\omega_p$. If $p \geq 1$ and $\log(w/\omega) \notin L^p$, then $L^\omega_p$ is a proper subset of $L^w_p$, and the topology defined on $L^\omega_p$ by the metric $\rho^\omega_p$ is strictly stronger than that induced on $L^\omega_p$ by the metric $\rho^\omega_p$. As an application, we show that if $\log w \in L^p$, then $(L_p, \rho^w_p)$ is an $F$-algebra, and the metric topologies $\rho^w_p$ and $\rho_p$ are the same. Finally, if $p \geq 1$ and $w$ is a weight such that $\log^+ w \in L^p$, we give four equivalent necessary and sufficient conditions for the space $(L_p, \rho^w_p)$ to be an $F$-algebra.
2. Equivalent metrics on the space $L_p^+(0 < p < \infty)$

Let $(\Omega, \Sigma, \mu)$ be a measure space, i.e. $\Omega$ is a nonempty set, $\Sigma$ is a $\sigma$-algebra of subsets of $\Omega$ and $\mu$ is a nonnegative finite, complete measure not vanishing identically. Denote by $L_p^+ (\mu) = L_p^+(0 < p \leq \infty)$ the familiar Lebesgue spaces on $\Omega$. For $p > 0$, we define the class $L_p^+(\mu) = L_p^+$ of all (equivalence classes of) $\Sigma$-measurable complex functions $f$ defined on $\Omega$ such that the function $\log^+ |f|$ is in $L_p$, where $\log^+ a = \max\{\log a, 0\}$, i.e. such that

$$\int_\Omega (\log^+ |f(x)|)^p d\mu < \infty.$$

Obviously, $L_q^+ \subset L_p^+$ for $q > p$, and from the inequality $\log^+ x \leq x^s/\se, x \leq 0, s > 0$, we see that $\bigcup_{p>0} L^p \subset \bigcap_{p>0} L^+_p$. Combining the inequalities $\log^+ |f+g| \leq \log^+ |f|+\log^+ |g|+\log 2$, $\log^+ |fg| \leq \log^+ |f|+\log^+ |g|$ with $(|x|+|y|)^p \leq 2^{\max\{p-1,0\}}(|x|^p+|y|^p)$ and $(|x|+|y|+|z|)^p \leq 3^{\max\{p-1,0\}}(|x|^p+|y|^p+|z|^p)$, respectively, we see that every space $L_p^+$ is an algebra with respect to the pointwise addition and multiplication.

For each $p > 0$, we define the metric $d_p$ on $L_p^+$ by

$$d_p(f,g) = \inf_{t>0} [t + \mu (\{x \in \Omega : |f(x) - g(x)| \geq t\})]$$

$$+ \int_\Omega |(\log^+ |f(x)|)^p - (\log^+ |g(x)|)^p| d\mu. \tag{2.1}$$

The space $L_1^+$ was introduced in [G, p. 122], with the notation $L(\mu)$ in [G]. In fact, the above metric $d_p$ with $p = 1$ coincides with the Gamelin-Lumer’s metric $d$ defined on $L_1^+$. It was proved in [G, p. 122, Theorem 2.3] that the space $L_1^+$ with the topology given by the metric $d_1$ is a topological algebra. The following result is a generalization of the corresponding result for the case $p = 1$. The proof of this result is completely analogous to those for the case $p = 1$ given in [G, p. 122], and therefore may be omitted.

**Theorem 2.1.** The space $L_p^+$ with the metric $d_p$ given by (1.2) is a topological algebra, i.e. a topological vector space with a complete metric in which multiplication is continuous.

By the inequality

$$\left( \log(1 + |x|) \right)^p \leq 2^{\max\{p-1,0\}} \left( \log 2 \right)^p + (\log^+ |x|)^p, \tag{2.2}$$
it follows that \( f \) belongs to \( L^+_p \) if and only if there holds
\[
\left( \| f \|_p \right)^p := \int_{\Omega} (\log(1 + |f(x)|))^p \, d\mu < \infty.
\]

By the inequality \( \log(1 + |f + g|) \leq \log(1 + |f|) + (\log(1 + |g|) \) and Minkowski’s inequality, it follows that for \( p > 1 \) the function \( \rho_p \), defined as
\[
(2.3) \quad \rho_p(f, g) = \| f - g \|_p, \quad f, g \in L^+_p,
\]
satisfies the triangle inequality. Combining the above inequality with \( (|x| + |y|)^p \leq |x|^p + |y|^p \) for \( 0 < p \leq 1 \), we see that the function \( \rho_p \) defined as
\[
\rho_p(f, g) = \left( \| f - g \|_p \right)^p, \quad f, g \in L^+_p, 0 < p \leq 1,
\]
satisfies also the triangle inequality. Hence, \( \rho_p \) is an invariant metric on \( L^+_p \) for all \( p > 0 \).

Recall that a subset \( K \) of \( L^p \) forms an uniformly integrable family if for given \( \varepsilon > 0 \) there exists a \( \delta > 0 \) so that
\[
\int_E |f(x)| \, d\mu < \delta \quad \text{for all} \quad f \in K,
\]
whenever \( E \subset \Omega \) with its measure \( \mu(E) < \delta \).

Two metrics (or norms) defined on the same space will be called equivalent if they induce the same topology.

For the proof of Theorem 2.3 we will need the following lemma.

**Lemma 2.2.** ([G, p. 122, Theorem 1.3]). Let \((f_n)\) be a sequence in \( L^1 \) and \( f \in L^1 \) such that \( f_n \to f \) in \( L^1 \). Then a sequence \((f_n)\) is a uniformly integrable family. Conversely, if a sequence \((f_n)\) is a uniformly integrable family on \( \Omega \), and \( f_n \to f \) in measure, then \( f \) belongs to \( L^1 \) and \( f_n \to f \) in \( L^1 \).

**Theorem 2.3.** The metric \( d_p \) defines a topology for \( L^+_p \) which is equivalent to the topology defined by the metric \( \rho_p \).

*Proof.* Suppose first that \( \rho_p(f_n, f) \to 0 \) as \( n \to \infty \), where \((f_n)\) and \( f \) are in \( L^+_p \). Then by Chebyshev’s inequality, it is easily seen that \( f_n \to f \) in measure on \( \Omega \). For simplicity, put
\[
\left( \| f \|_E \right)^{\max\{p, 1\}} = \int_E (\log(1 + |f(x)|))^p \, d\mu
\]
for any measurable set $E \subset \Omega$. By the triangle inequality and (2.2), we have

$$
\|f_n\|_E \leq \|f_n - f\|_E + \|f\|_E
$$

(2.4)

\[
\leq \rho_p(f_n, f) + 2^{\max\{1-1/p, 1\}} \times \left( \mu(E) (\log 2)^p + \int_E (\log^+ |f(x)|)^p \, d\mu \right)^{1/\max\{p, 1\}}.
\]

This shows that \( \{ (\log^+ |f_n(x)|)^p : n \in \mathbb{N} \} \) form a uniformly integrable family, and by Lemma 2.2, \( \rho_p(f_n, f) \to 0 \) as \( n \to \infty \).

Conversely, assume that \( d_p(f_n, f) \to 0 \) as \( n \to \infty \). Then, by the definition of \( d_p \), \( f_n \to f \) in measure, and by Lemma 2.2, \( \{ (\log^+ |f_n(x)|)^p : n \in \mathbb{N} \} \) are uniformly integrable. Replacing in (2.3) \( f \) by \( f_n \) and \( f - f_n \), we see that the family \( \{ (\log(1 + |f_n(x) - f(x)|))^p : n \in \mathbb{N} \} \) is uniformly integrable. Thus, by Lemma 2.2, \( \rho_p(f_n, f) \to 0 \) as \( n \to \infty \). Hence the metrics \( d_p \) and \( \rho_p \) are equivalent.

Remark. Using the same argument applied in the proof of Theorem 2.3, it is easy to see that the metrics \( \rho_p \) and \( d_p \) are equivalent with the metric \( \delta_p \) given on \( L^+_p \) by

\[
\delta_p(f, g) = \inf_{t > 0} \left[ t + \mu(\{x \in \Omega : |f(x) - g(x)| \geq t\}) \right]
\]

\[
+ \left( \int_\Omega \left| \log^+ |f(x)| - \log^+ |g(x)| \right|^p \, d\mu \right)^{1/\max\{p, 1\}}, f, g \in L^+_p.
\]

Remark. In [Y, Remark 5, p. 460], M. Hasumi pointed out that the Yanagihara’s metric \( \rho = \rho_1 \) defines a topology for the space \( L^+_1 = L(\mu) \), which is equivalent to the metric topology \( d_1 = d \) used by Gamelin-Lumer in [G, p. 122].

Corollary 2.4. The space \( L^+_p, p > 0 \), with the topology given by the metric \( \rho_p \), is an \( F \)-algebra, i.e. a topological algebra with a complete translation invariant metric \( \rho_p \).

Proof. We see from Theorems 2.1 and 2.3 that it is sufficient to show that the space \( L^+_p \) is complete with respect to the metric \( \rho_p \). The completeness of \( L^+_p \) may be proved by the standard manner as the completeness of the Lebesgue spaces \( L^p \) or an arbitrary generalized Orlicz space with a corresponding \( F \)-norm (for example, see the proof of Theorem 7.7 in [Mu, p. 35]). In view of this, note that \( L^+_p \) may be
considered as the generalized Orlicz space $L^w_p$ with a constant function $w(t) \equiv 1$ on $[0, 2\pi)$ (see Section 6).

**Theorem 2.5.** If $0 < p < s < \infty$, then $L^s_+ \subset L^p_+$, and the metric $d_p$ induces a topology for $L^s_+$ which is coarser than the topology defined on $L^s_+$ by the initial metric $d_s$.

*Proof.* If $0 < p < s < \infty$, $L^s_+ \subset L^p_+$ is obvious. Let $(f_n)$ be a sequence in $L^s_+$ and $f \in L^s_+$ such that $d_s(f_n, f) \to 0$ as $n \to \infty$. This means by definition of $d_s$, that $(\log^+ |f_n|)^s \to (\log^+ |f|)^s$ in $L^1$, and $f_n \to f$ in measure. Then by Lemma 2.2, a sequence $((\log^+ |f_n|)^s)$ is a uniformly integrable family on $\Omega$. Since $0 < p < s < \infty$, it is routine to verify that $((\log^+ |f_n|)^p)$ is also a uniformly integrable family on $\Omega$. Then by Lemma 2.2, $(\log^+ |f_n|)^p \to (\log^+ |f|)^p$ in $L^1$, and hence $d_p(f_n, f) \to 0$ as $n \to \infty$. □

3. A generalization of Hardy algebras

Let $A$ be a uniform algebra on a compact Hausdorff space $\Omega$. Let $\mu$ be a representing measure for a nonzero complex-valued homomorphism $\alpha$ of $A$. For $0 < q \leq \infty$, denote by $H^q(\mu)$ the Hardy space (the closure of $A$ in the Lebesgue space $L^q = L^q(\mu)$). Motivated by the definition of the Hardy algebra $H(\mu)$ given in [G, p. 123], here we define the space $H^p(\mu) = H_p(0 < p < \infty)$, as the $d_p$-closure of $A$ in the space $L^p_+ = L^p_+$ defined in the Section 2. Observe that the space $H_1$ coincides with the Hardy algebra $H(\mu)$ defined in [G, p. 122]. From Theorem 2.5 we obtain immediately its $H^p$-analogue.

**Theorem 3.1.** If $0 < p < s < \infty$, then $H^s \subset H^p$, and the metric $d_p$ induces a topology for $H^s$ which is coarser than the topology defined on $H^s$ by the initial metric $d_s$.

Denote by $M_\alpha$ the set of all representing measures for a homomorphism $\alpha$. For the proof of Theorem 3.2, we will need the following lemma which generalizes Corollary 2.2 in [G, p. 122].

**Lemma 3.2.** For all $0 < q \leq \infty$ and all $p > 0$, the topology of $L^q$ is stronger than the topology of the space $L^p_+$.

*Proof.* Clearly, it suffices to consider the case $0 < q \leq 1$. Let $(f_n)$ be a sequence in $L^q$ such that $f_n \to f$ in $L^q$ for some $f \in L^q$. From the inequality $\log(1 + x) \leq x^s/s, x \geq 0, 0 < s \leq 1$, it follows that $(\log(1 + x))^p \leq px^q/q$, and hence $\rho_p(f_n, f) \to 0$ as $n \to \infty$. Therefore, we conclude by Theorem 2.3 that $d_p(f_n, f) \to 0$ as $n \to \infty$, which completes the proof. □
Theorem 3.3. Suppose that the space $M_\alpha$ is finite-dimensional, and let $\mu$ be a core representing measure for a homomorphism $\alpha$. Then for any fixed $0 < q \leq \infty$, there holds

\[
\bigcap_{p>0} H_p \cap L^q = H^q(\mu).
\]

Furthermore, there holds

\[
H_p \cap L^q = H^q(\mu) \quad \text{for } 1 \leq p \leq q \leq \infty.
\]

Proof. By [G, p. 125, Theorem 4.2], for all $0 < q \leq \infty$ we have $H_1 \cap L^q = H^q(\mu)$. This shows that $\bigcap_{p>0} H_p \cap L^q \subseteq H^q(\mu)$.

Suppose now that $f \in H^q(\mu)$ for some $0 < q \leq \infty$. Then there is a sequence $(f_n)$ in $A$ such that $f_n \to f$ in $L^q$ as $n \to \infty$. By Lemma 3.2, $f_n \to f$ in $L^+_p$ as $n \to \infty$, for each $p > 0$. Thus, $f \in \bigcap_{p>0} H_p$, which implies that $H^q(\mu) \subseteq \bigcap_{p>0} H_p \cap L^q$. This yields (3.1). If $1 \leq p \leq q \leq \infty$, by Theorem (3.1) we have

\[
H_p \cap L^q \subseteq H_1 \cap L^q = H^q(\mu).
\]

Conversely, from (3.1) we obtain $H^q(\mu) = \bigcap_{s>0} H_s \cap L^q \subseteq H_p \cap L^q$. This proves (3.2). \hfill \Box

4. ORLICZ SPACES $L^+_p(dt/2\pi)(0 < p < \infty)$

The function $\psi : [0, \infty) \mapsto [0, \infty)$ defined as $\psi(t) = (\log(1 + t))^p$, is continuous and nondecreasing in $[0, \infty)$, such that $\psi(0) = 0$, $\psi(t) > 0$ for $t > 0$, and $\lim_{t \to +\infty} \psi(t) = +\infty$, is called an Orlicz function or a $\varphi$-function (see [Mu, p. 4, Examples 1.9]). Further, observe that the space $L^+_p(dt/2\pi) = L_p$, $p > 0$, consisting of all complex-valued functions $f$, defined and measurable on $[0, 2\pi)$, for which

\[
\left(\|f\|_p\right)^p := \int_0^{2\pi} (\log(1 + |f(t)|))^p \frac{dt}{2\pi} < +\infty.
\]

is the Orlicz class (see [Mu, p. 5]), whose generalization we give in Section 6. It follows by the dominated convergence theorem that the class $L_p$ coincides with the associated Orlicz space (see [Mu, p. 2, Definition 1.4]), consisting of those functions $f \in L_p$ such that

\[
\int_0^{2\pi} (\log(1 + c|f(t)|))^p \frac{dt}{2\pi} \to 0 \quad \text{as} \quad c \to 0 +.
\]
Furthermore, since by $\rho_p(f,g) = (\| f - g \|_p)^{\min\{p,1\}}$, $f, g \in L_p$, is defined an invariant metric on $L_p$, the function $\| \cdot \|_p$ given by 4.1 is a modular in the sense of Definition 1.1 in [Mu, p. 1]. For any fixed $f \in L_p$, by the monotone convergence theorem, we see that $\lim_{c \to 0} \psi(cf) \to 0$, and so $(L_p, \rho_p)$ is a modular space in the sense of Definition 1.4 in [Mu, p. 2]. In other words, the function $\| \cdot \|_p$ is an $F$-norm. It is well known (see [Mu, Theorem 1.5, p. 2 and Theorem 7.7, p. 35]) that the functional $| \cdot |_p$ defined as

$$|f|_p = \inf \left\{ \varepsilon > 0 : \int_0^{2\pi} \left( \log \left(1 + \frac{|f(t)|}{\varepsilon} \right) \right)^p \frac{dt}{2\pi} \leq \varepsilon \right\}, f \in L_p,$$

is a complete $F$-norm. Furthermore (see [L, p. 54]), $L_p$ is a completion (closure) of the space of all continuous functions on $[0, 2\pi]$ in the space $(L_p, \| \cdot \|_p)$.

**Theorem 4.1.** The $F$-norms $\| \cdot \|_p$ and $| \cdot |_p$ induce the same topology on the space $L_p$. In other words, the norm and modular convergences are equivalent.

**Proof.** Since a $\varphi$-function $\psi(t) = (\log(1 + t))^p$ satisfies obviously the so-called $\Delta_2$-condition given by

$$(\Delta_2) \quad \phi(2t) \leq c\phi(t)$$

with a constant $c = 2^p$, the assertion follows immediately from [L, p. 55, 2.4]. But, we give here a direct proof.

Let $K_\rho(0,r) = \{ f \in L_p : \| f \|_p < r \}$ be a neighborhood of 0 in the space $(L_p, \| \cdot \|_p)$, and let $f \in K_\rho(0,r)$ be any fixed. Consider a neighborhood $K_\psi(f,\varepsilon/2) = \{ f \in L_p : \| f \|_p < r \}$ of $f$ in the space $(L_p, | \cdot |_p)$, and assume $g \in K_\psi(f,\varepsilon/2)$. Then we have

$$\frac{\varepsilon}{2} > |g - f|_p = \inf \{ \delta > 0 : \frac{\| g - f \|}{\delta} < \delta \} = \sigma.$$

Thus, from $\varepsilon/2 > \sigma$ we infer that

$$\frac{\| g - f \|}{\varepsilon/2} < \frac{\varepsilon}{2},$$
whence we obtain
\[ \|g\|_p \leq \|g - f\|_p + \|f\|_p \]
\[ \leq \left\| \frac{g - f}{\varepsilon/2} \right\| + \frac{\varepsilon}{2} + r - \varepsilon \]
\[ < \frac{\varepsilon}{2} + \log\left(1 + \frac{\varepsilon}{2}\right) + r - \varepsilon \]
\[ < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + r - \varepsilon = r. \]
This shows that \( K_\psi(f, \varepsilon/2) \subset K_\rho(0, r) \).

Conversely, let \( K_\psi(0, r) \) be a neighborhood of 0 in the space \( (L_p, |\cdot|_p) \), and let \( f \in K_\psi(0, r) \) be any fixed. Put \( \sigma = r - |f|_p \), take \( k \) an integer so that \( k > \frac{2}{\sigma} \), and set \( \varepsilon = \frac{\sigma}{2}k \). Then for all \( g \in K_\rho(f, \varepsilon) \) by the triangle inequality, we have
\[ \left\| \frac{g - f}{\sigma/2} \right\| < k\|g - f\|_p \leq k\varepsilon = \frac{\sigma}{2}. \]
Hence, we obtain \( |g - f|_p \leq \sigma/2, \) and so
\[ |g|_p \leq |g - f|_p + |f|_p \leq \frac{\sigma}{2} + r - \sigma = r - \frac{\sigma}{2} < r. \]
Thus, \( g \in K_\psi(0, r) \), i.e. \( K_\rho(f, \varepsilon) \subset K_\psi(0, r) \). Hence, the topologies defined on the space \( L_p \) by \( \| \cdot \|_p \) and \( | \cdot |_p \) are the same. \( \square \)

As an application of Theorem 4.1, we obtain the following result.

**Corollary 4.2.** There does not exist a nontrivial continuous linear functional on the space \( (L_p, \| \cdot \|_p) \).

**Proof.** Since the Orlicz function \( \psi(t) = (\log(1 + t))^p \) satisfies the condition \( \liminf_{t \to \infty} t^{-1} \psi(t) = 0 \), it follows by [MO] that there does not exist a nontrivial modular continuous linear functional on the space \( (L_p, | \cdot |_p) \). Hence, this is true by Theorem 4.1 for the space \( (L_p, \| \cdot \|_p) \). \( \square \)

**5. Privalov’s spaces \( N^p(1 < p < \infty) \)**

As in [G, p. 125], the Hardy algebra \( H_1(d\theta/2\pi) \) consists of all functions \( f \) holomorphic on the unit disk \( D : |z| < 1 \) for which the family
\[ \\{ \log^+ |f(re^{i\theta})| : 0 \leq r < 1 \} \]
is uniformly integrable on the unit circle \( T \), that is, for a given \( \varepsilon > 0 \), there exists \( \delta > 0 \) so that
\[ \int_E \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi} < \varepsilon, \quad 0 \leq r < 1, \]
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for any measurable set \( E \subset [0, 2\pi] \) with its Lebesgue measure \( |E| < \delta \). This space is usually called the Smirnov class \( N^+ \). For \( p > 1 \), following I. I. Privalov (see [P, p. 93], where \( N^p \) is denoted as \( A_q \)), a function \( f \) holomorphic in \( D \), belongs to the class \( N^p \), if there holds

\[
\sup_{0 \leq r < 1} \int_0^{2\pi} (\log^+ |f(re^{i\theta})|)^p \frac{d\theta}{2\pi} < \infty,
\]

For \( p = 1 \), the condition (5.1) defines the Nevanlinna class \( N \) of holomorphic functions in \( D \). Recall that for \( f \in N \), the radial limit

\[
f^*(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})
\]

exists for almost every \( e^{i\theta} \) and \( \log|f^*| \in L^1 \) unless \( f \not\equiv 0 \). It is known (see [Mo]) that

\[
\bigcup_{p>1} N^p \subset N^+ \subset N.
\]

Observe that the algebra \( N^p \) may be considered as the Hardy-Orlicz space with the Orlicz function \( \psi : [0, \infty) \mapsto [0, \infty) \) defined as \( \psi(t) = (\log(1 + t))^p \). For more informations on the Hardy-Orlicz spaces see [Mu, Ch. IV, Sec. 20]. Identifying a function \( f \in N \) with its boundary function \( f^* \), by [L, 3.4, p. 57], the space \( N^p \) is identical with the closure of the space of all functions holomorphic in \( D \) and continuous in \( \overline{D} : |z| \leq 1 \) in the space \( (L^p(dt/2\pi) \cap N, |\cdot|_p) \). On the other hand, \( (N^p, d_p) \) coincides with the space \( H^p(d\theta/2\pi) \) defined in Section 3, i.e. with the closure of the disk algebra \( P(D) \) in the space \( (L^p(dt/2\pi), d_p) \). Therefore, the space \( N^p \) may be considered as generalized Hardy algebra. From this fact, Theorem 2.3 and Corollary 2.4, it follows immediately the following result of M. Stoll obtained in [S].

**Theorem 5.1.** ([S, Theorem 4.2]). For \( p > 1 \), the space \( N^p \) with the topology given by the metric \( \rho_p \) is an \( F \)-algebra. Furthermore, the polynomials are dense in \( (N^p, \rho_p) \), and hence \( N^p \) is separable.

6. **THE GENERALIZED ORLICZ SPACES** \( L^w_p \)

Let \( (\Omega, \Sigma, \mu) \) be a measure space, and let \( w \) be a \( \Sigma \)-measurable non-negative function defined on \( \Omega \) such that \( w(t) > 0 \) \( \mu \)-almost every on \( \Omega \). For \( p > 0 \), the real function \( \psi_w \) defined on \( \Omega \times [0, \infty) \) by the formula

\[
\psi_w(t, u) = (\log(1 + uw(t)))^p
\]

is a Musielak-Orlicz function, since it satisfies the Caratheodory conditions, i.e. it satisfies the following conditions (see [Mu, p. 33, Definition 7.1]):
(i) \( u \mapsto \psi_w(t, u) \) is a \( \varphi \)-function of the variable \( u \geq 0 \) for \( \mu \)-almost every \( t \), i.e. is a nondecreasing, continuous function of \( u \) such that \( \psi_w(t, 0) = 0, \psi_w(t, u) > 0 \) for \( u > 0 \), \( \lim_{u \to \infty} \psi_w(t, u) \to \infty \); (ii) \( \psi_w(t, u) \) is a \( \Sigma \)-measurable function of \( t \) for all \( u \geq 0 \).

For the Lebesgue’s measure space \((\Omega, [0, 2\pi], dt/2\pi))\), denote by

\[
L_w^p(d t/2\pi) = L_w^p, \quad p > 0, \quad \text{the class of all (equivalence classes of)} \quad \text{complex-valued functions } f, \text{ defined and measurable on } [0, 2\pi),
\]

\[
\| f \|_{w^p} := \frac{2\pi}{2\pi} \int_0^{2\pi} (\log(1 + |f(t)| w(t)))^p \, \frac{dt}{2\pi} < +\infty.
\]

\( L_w^p \) is called the generalized Orlicz class with the modular \( \| \cdot \|_{w^p} \) (see [Mu, p. 33, Definition 7.2]). It follows by the dominated convergence theorem that the class \( L_w^p \) coincides with the associated generalized Orlicz space consisting of those functions \( f \in L_w^p \) for which

\[
\lim_{c \to 0^+} \frac{2\pi}{2\pi} \int_0^{2\pi} (\log(1 + c |f(t)| w(t)))^p \, \frac{dt}{2\pi} \to 0.
\]

Furthermore, from the inequality \( \| cf \|_{w^p} \leq \| c \|_{w^p} \| f \|_{w^p} \), we see that \( L_w^p \) coincides with the space of all finite elements of \( L_w^p \) consisting of those functions \( f \in L_w^p \) such that \( cf \in L_w^p \) for every \( c > 0 \). Since \( w(t) > 0 \) almost every on \([0, 2\pi)\) and

\[
\| f + g \|_{w^p} = \| (f + g)w \|_p \leq \| fw \|_p + \| gw \|_p = \| f \|_{w^p} + \| g \|_{w^p},
\]

\( \| \cdot \|_{w^p} \) is a modular in the sense of Definition 1.1 of [Mu, p. 1]. In other words, the function \( \| \cdot \|_{w^p} \) is an \( F \)-norm. Hence, by \( \rho^w_p(f, g) = (\| f - g \|_{w^p})_{\min\{p, 1\}} \), \( f, g \in L_w^p \), is defined an invariant metric on \( L_w^p \). By [Mu, p. 2, Theorem 1.5, and p. 35, Theorem 7.7], it follows that the functional \( | \cdot |_{w^p} \) defined for \( f \in L_w^p \) as

\[
| f |_{w^p} = \inf \left\{ \varepsilon > 0 : \frac{2\pi}{2\pi} \int_0^{2\pi} \left( \log \left( 1 + \frac{|f(t)| w(t)}{\varepsilon} \right) \right)^p \, \frac{dt}{2\pi} \leq \varepsilon \right\},
\]

is a complete \( F \)-norm.

**Theorem 6.1.** \((L_w^p, | \cdot |_{w^p})\) is an \( F \)-space. Furthermore, \( F \)-norms \( \| \cdot \|_{w^p} \) and \( | \cdot |_{w^p} \) induce the same topology on the space \( L_w^p \).

**Proof.** Since the functional \( | \cdot |_{w^p} \) is a complete \( F \)-norm, and hence for a fixed \( f \in L_w^p \), \( c \mapsto cf \) is a continuous mapping from \( C \) into \( L_w^p \), it suffices (see [DS, p. 51]) to check the continuity of the mapping \( f \mapsto cf \).
from \( L^w_p \) into \( L^w_p \), for a fixed \( c \in \mathbb{C} \). Take \( k \) an integer with \( |c| \leq k \). Then

\[
|cf| \leq |kf| \leq k|f|
\]

so \( f \mapsto cf \) is continuous.

The second assertion of the theorem can be proved completely analogously as Theorem 4.1. \( \square \)

**Theorem 6.2.** For any weight \( w \), \( (L^w_p, \rho^w_p) \) is an \( F \)-space.

**Proof.** To show that \( (L^w_p, \rho^w_p) \) is an \( F \)-space, it suffices by Theorem 6.1 to prove that \( (L^w_p, \rho^w_p) \) is complete. Let \( (f_n) \) be a Cauchy sequence in \( (L^w_p, \rho^w_p) \). This means that \( (wf_n, g) \) is a Cauchy sequence in \( (L^w_p, \rho^w_p) \), and by Corollary 2.4, there is a \( g \in L^w_p \) such that \( \rho_p(wf_n, g) \to 0 \) as \( n \to \infty \).

Then \( \rho^w_p(f_n, g/\omega) = \rho_p(wf_n, g) \to 0 \), and since \( \rho^w_p(g/\omega, 0) = \rho_p(g, 0) < \infty \), we see that \( f = g/\omega \in L^w_p \). Hence, the space \( (L^w_p, \rho^w_p) \) is complete.

**Theorem 6.3.** Let \( w \) and \( \omega \) be two weights such that \( \log^+ (w/\omega) \in L^p \).

Then:

(i) \( L^w_p \subset L^w_p \), and \( L^w_p = L^w_p \) if and only if \( \log (w/\omega) \in L^p \).

(ii) The topology defined on \( L^w_p \) by the metric \( \rho^w_p \) is stronger than that induced on \( L^w_p \) by the metric \( \rho^w_p \).

(iii) Let \( p \geq 1 \). If \( \log (w/\omega) \notin L^p \), then \( L^w_p \) is a proper subset of \( L^w_p \).

Furthermore, the topology defined on \( L^w_p \) by the metric \( \rho^w_p \) is strictly stronger than that induced on \( L^w_p \) by the metric \( \rho^w_p \).

(iv) Let \( p \geq 1 \). If \( (L^w_p, \rho^w_p) \) with \( p \geq 1 \) is a complete metric space, then \( \log (w/\omega) \in L^p \), and the topology defined on \( L^w_p = L^w_p \) by the metric \( \rho^w_p \) coincides with that induced on \( L^w_p \) by the metric \( \rho^w_p \).

**Proof.** (i) The inclusion relation \( L^w_p \subset L^w_p \) follows immediately from the inequality \( ||f||^w_p = ||f\omega w/\omega||^p_p \leq ||f\omega||^p_p + ||w/\omega||^p_p ||f||^p_p + ||w/\omega||^p_p = ||f\omega w/\omega||^p_p \), and the fact that \( w/\omega \in L^p \). If \( L^w_p = L^w_p \), then since \( 1/\omega \in L^w_p \), it follows that \( \log^+ (w/\omega) \in L^p \), and so \( \log (w/\omega) \in L^p \).

Conversely, if \( \log (w/\omega) \in L^p \), then \( \log^+ (w/\omega) \in L^p \), and thus \( L^w_p \subset L^w_p \), which implies \( L^w_p = L^w_p \).

(ii) Assume that \( (f_n) \) is a sequence in \( L^w_p \) and \( f \in L^w_p \) such that \( f_n \to f \) in \( (L^w_p, \rho^w_p) \) as \( n \to \infty \). This means that \( \rho^w_p(f_n, f) \to 0 \), or equivalently, \( \rho_p(wf_n, \omega f) \to 0 \) as \( n \to \infty \). Since \( w/\omega \in L^p \), by Corollary 2.4, we have \( \rho_p(wf_n, w f) \to 0 \), as \( n \to \infty \). Thus, \( \rho^w_p(f_n, f) \to 0 \) as \( n \to \infty \).

(iii) If \( \log (w/\omega) \notin L^p \), the strict inclusion relation \( L^w_p \subset L^w_p \) follows from the assertion (i). Since \( \log^+ (w/\omega) \in L^p \) and \( \log (w/\omega) \notin L^1 \), by
[MP, Theorem 3.1], there holds
\[
\inf_{P \in P_0} \frac{2\pi}{\int_0^{2\pi} \left( \log \left( 1 + |P(e^{it})| \right) \right)^p \, dt}{2\pi} = 0,
\]
or equivalently, that there exists a sequence \((P_n)\) of (analytic) polynomials with \(P_n(0) = 1\) such that \(\rho_p^w(P_n \omega, 0) \to 0\) as \(n \to \infty\). On the other hand, by the same theorem, there holds
\[
\inf_{P \in P_0} \frac{2\pi}{\int_0^{2\pi} \left( \log \left( 1 + |P(e^{it})| \right) \right)^p \, dt}{2\pi} = (\log 2)^p,
\]
and hence the above sequence \((P_n)\) does not converge to 0 in the space \((L_p^w, \rho_p^w)\), or equivalently that a sequence \((P_n / \omega)\) does not converge to 0 in the space \((L_\omega^w, \rho_\omega^p)\). Hence, the metric topology \(\rho_\omega^p\) is strictly stronger than the metric topology \(\rho_p^w\).

(iv) Suppose that \((L_\omega^w, \rho_p^w)\) is a complete metric space, but \(\log (w / \omega) \in L^p \setminus L^1\). We will consider two cases.

Case 1. \(\log (w / \omega) \notin L^1\). By Theorem 6.2, the space \((L_\omega^w, \rho_\omega^p)\) is an \(F\)-space, and by the assumption, \((L_p^w, \rho_p^w)\) is complete, and henceforth it is an \(F\)-space. We see from (iii) that the topology defined on \(L^\omega_p\) by the metric \(\rho_p^w\) is strictly stronger than that induced on \(L^\omega_p\) by the metric \(\rho_p^w\). Consider the identity map
\[j : (L_\omega^w, \rho_p^w) \to (L_\omega^w, \rho_p^w)\]
Then by (ii), \(j\) is continuous. By the open mapping theorem, we conclude that the inverse \(j^{-1}\) of \(j\) is also continuous. This shows that the metric topologies \(\rho_p^w\) and \(\rho_p^w\) must be the same. A contradiction.

Case 2. \(\log (w / \omega) \in L^1\). Since \(\log^+ (\omega / w) \in L^p \subset L^1\), it follows that \(\log^+ (\omega / w) \notin L^1\), and hence \(1/w \in L_p^w \setminus L_\omega^w\). Define the outer function \(F\) by
\[
F(z) = \exp \left( \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log \frac{w(t)}{\omega(t)} \, dt \right) \frac{dt}{2\pi}, \quad z \in D.
\]
Then \(|F^*(e^{it})| = w(t) / \omega(t)\) at almost every \(t \in [0, 2\pi]\). If \(p = 1\), by the canonical factorization theorem for the Smirnov class \(N^+\) (see [D, Theorem 2.10]), it follows that \(F\) belongs to \(N^+\). Similarly, if \(p > 1\), by the canonical factorization theorem for the Privalov’s spaces \(N^p\) (see [P, p. 98]), it follows that \(F\) belongs to \(N^p\). For simplicity, we shall write \(N^1\) instead of \(N^+\). By a result of Mochizuki (see [Mo, Theorem 2]), there is a sequence \((f_n)\) in \(N^p\), \(p > 1\), such that \(f_n F \to 1\) in \(N^p\) as \(n \to \infty\). For \(p = 1\), put \(f_n = 1/F \in N^1\) for all \(n\). Hence, in both cases
Then the following statements about \( w \) and \( \rho_j \).

Consider the identity map \( F \) on \( \mathbb{R} \).

By Corollary 6.4, \( j \) is continuous. Since by Theorem 6.2 \( (L_p, \rho_p) \) is an \( F \)-space, by the open mapping theorem, we conclude that the inverse \( j^{-1} \) of \( j \) is also continuous. This shows that the metric topologies \( \rho_p \) and \( \rho^w_p \) must be the same on \( L_p \).

**Theorem 6.7.** Let \( p \geq 1 \) and let \( w \) be a weight such that \( \log^+ w \in L^p \).

Then the following statements about \( w \) are equivalent.
(i) \( \log w \in L^p \);
(ii) \( L^w_p = L_p \);
(iii) The metrics \( \rho^w_p \) and \( \rho_p \) define the same topology on \( L_p \);
(iv) \((L_p, \rho^w_p)\) is a complete metric space. (v) \((L_p, \rho_p)\) is an F-algebra;

**Proof.** (i) \( \Leftrightarrow \) (ii). Follows from Corollary 6.4.

(iii) \( \Rightarrow \) (i). This is a consequence of the assertion (iii) of Theorem 6.3, by setting \( \omega(t) \equiv 1 \) for \( t \in [0, 2\pi) \).

(i) \( \Rightarrow \) (iii). This is immediate from Corollary 6.4.

(iv) \( \Rightarrow \) (i). This follows from the assertion (iv) of Theorem 6.3, by setting \( \omega(t) \equiv 1 \) for \( t \in [0, 2\pi) \).

(i) \( \Rightarrow \) (v). This is immediate from Corollary 6.5.

(v) \( \Rightarrow \) (iv). This is obvious. \( \Box \)

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