REMARKS ON SOME CLASSES OF HOLOMORPHIC FUNCTIONS∗

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Abstract. Subclasses $N^q$ $(1 < q < \infty)$ of the Nevanlinna class $N$ are characterized. The containment relations with other classes are given. Consequently, we give a criterion for a holomorphic function to be in the space $N^q$. The canonical factorization theorem for elements of $N^q$ is proved.

1. Introduction

The class $N^q$ $(1 < q < \infty)$ consists of all holomorphic functions $f$ on the open unit disc in the complex plane which satisfy

$$\sup_{0 \leq r < 1} \int_0^{2\pi} \left( \log^+ |f(re^{i\theta})| \right)^q \frac{d\theta}{2\pi} < \infty. \quad (1)$$

These classes are introduced in the first edition of Privalov’s book [4]. In Section 3, we give the inclusion relations between $N^q$ and some other classes of holomorphic functions. Theorem 4.1 and the canonical factorizations, described by 2.2(a) and (b), show that the Smirnov class $N^+$ may be considered as the ’natural limit’ of classes $N^q$ as $q \to 1$. In the next section, the criteria for belonging to the classes $N^q$ are given. Finally, in Section 5, we give a new proof of the Canonical factorization theorem 2.2(b) given in [4].

2. Preliminary notations, definitions and results

Let $D$ denote the unit disc $|z| < 1$ in the complex plane, and let $T$ denote the boundary of $D$. Let $d\mu = d\theta/2\pi$ be the usual Lebesgue measure on $T$, and let $L^p = L^p(\mu)$ $(0 < p \leq \infty)$ be the familiar Lebesgue spaces on the unit circle $T$.

1991 Mathematics Subject Classification. Primary 30H05, 46J15.
Fix $q > 1$. Following I. I. Privalov (see [4], p. 93, where $N^q$ is denoted as $A^q$), a holomorphic function $f(z)$ in $D$ belongs to the class $N^q$, if there holds

$$(2.1) \sup_{0 \leq r < 1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi} < \infty,$$

where $\log^+ a = \max(\log a, 0)$, $a \geq 0$, and $(\log^+ a)^q = (\log^+ a)^q$.

For $q = 1$, the condition (2.1) defines the Nevanlinna class $N$ of holomorphic functions in $D$. The Smirnov class $N^+$ consists of those functions $f \in N$ for which the family $\log^+ |f(re^{i\theta})|$ $(0 \leq r < 1)$, is uniformly integrable on the unit circle $T$, that is, for a given $\varepsilon > 0$, there exists $\delta > 0$ so that

$$\int_E \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi} < \varepsilon, \quad 0 \leq r < 1,$$

for any measurable set $E \subset T$ with $\mu(E) < \delta$.

Recall that the Hardy space $H^p$ $(0 < p \leq \infty)$ consists of all functions $f$, holomorphic in $D$, which satisfy

$$\sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} < \infty$$

if $0 < p < \infty$, and which are bounded when $p = \infty$:

$$\sup_{z \in D} |f(z)| < \infty.$$

As in [2], we denote by $M$ the class of all functions $f$ holomorphic in $D$, such that

$$\int_0^{2\pi} \log^+ Mf(\theta) \frac{d\theta}{2\pi} < \infty,$$

where

$$Mf(\theta) = \sup_{0 \leq r < 1} |f(re^{i\theta})|.$$

The study on the classes $N$, $N^+$ and $H^p$ has been well established (see [1], [3], [4]), as well as on the class $M$ by Hong Oh Kim [2]. Since in the second edition of [4], the part concerning the theory of $N^q$ spaces is not included, we summarize some facts from [4] which will be needed in the sequel. All of them, except 2.2(c), are proved in [4], pp. 79–101, where $N$, $N^+$ and $N^q$ are denoted as $A$, $B$ and $A^q$, respectively.
2.1. Radial limits. For \( f \in N \), the radial limit
\[
f^\ast(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})
\]
exists for almost every \( e^{i\theta} \) and \( \log |f^\ast| \in L^1 \) unless \( f \not\equiv 0 \).

2.2. Canonical factorization. A function \( f \in N \) can be factored as follows
\[
f(z) = B(z) \left( \frac{S_1(z)}{S_2(z)} \right) F(z),
\]
where \( B(z) \) is the Blaschke product with respect to zeros of \( f(z) \), \( S_k(z) \), \( k = 1, 2, \) are the singular inner functions with no common factor and \( F(z) \) is an outer function for the class \( N \), i.e.
\[
S_k(z) = \exp \left( -\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \, d\mu_k(t) \right)
\]
with positive singular measure \( d\mu_k \), \( k = 1, 2 \), and
\[
F(z) = \omega \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log |f^\ast(e^{it})| \, dt \right),
\]
where \( \omega \) is a constant of unit modulus.

Then we have the following statements about a function \( f \in N \) with the above factorization.

(a) \( f \) belongs to \( N^+ \) if and only if \( S_2 \equiv 0 \).
(b) \( f \) belongs to \( N^q \) if and only if \( S_2 \equiv 0 \) and \( \log^+ |f^\ast| \in L^1 \).
(c) \( f \) belongs to \( M \) if \( S_2 \equiv 0 \) and \( \log^+ |f^\ast| \in \text{Re } H^1 \). The converse is false (see [2], Theorem 2.2). \( \text{Re } H^1 \) denotes the class of all real parts of the class \( H^1 \), where \( H^1 \) is considered as a space of functions on \( T \) (see [2], [3]).
(d) \( f \) belongs to \( H^p \) \( (0 < p < \infty) \) if and only if \( S_2 \equiv 0 \) and \( |f^\ast| \in L^p \).

2.3. Privalov’s theorem. A function \( f(z) \) holomorphic in \( D \) belongs to the class \( N^q \) if and only if for given \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that
\[
\int_E \log^+ |f(re^{i\theta})| \, \frac{d\theta}{2\pi} < \varepsilon, \quad 0 \leq r < 1,
\]
for any measurable set \( E \subset T \), with \( \mu(E) < \delta \), i.e. if and only if \( \log^+ |f(re^{i\theta})| \) \( (0 \leq r < 1) \) form a uniformly integrable family.
3. THE INCLUSIONS AMONG THE VARIOUS CLASSES

Theorem 3.1. \( \cup_{q>1} N^q \subset M \subset N^+ \subset N \),
(ii) \( \cup_{s>0} H^s \subset \cap_{q>1} N^q \),
(iii) \( \cup_{q>p} N^q \subset \cap_{1<q<p} N^q \),
for each \( p > 1 \). The above inclusion relations are proper.

Proof of (i). The relation \( M \subseteq N^+ \subseteq N \) is proved in ([2], Theorem 2.1). Let \( f \in N^q \), for some \( q > 1 \). For the proof of the inclusion \( N^q \subseteq M \), by 2.2 (b) and (c), it suffices to show that \( h = \log^+ |f^*| \) belongs to the class \( \text{Re} H^1 \). By Theorem 1.5 of [2], this is equivalent to the condition \( h \in L \log L \), or \( h \log^+ h \in L^1 \), where \( L \log L \) denotes the Zygmund class (see also [3], pp. 135–136). Since \( h \in L^1 \), using the inequality \( \log^+ x \leq x^{\alpha/\alpha} \), \( x \geq 0, \alpha > 0 \), we have

\[
\int_0^{2\pi} \psi(t) \log^+ h(t) \, dt \leq \frac{2}{q-1} \left( \frac{q-1}{2} \int_0^{2\pi} \psi(t) \, dt \right) \in L^1.
\]

Thus, \( h \in L \log L \), which implies that \( \cup_{q>1} N^q \subseteq M \). Now we consider the function

\[
f(z) = \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \psi(t) \, dt \right),
\]

where \( \psi(t) \) is a step function defined by

\[
\psi(t) = \begin{cases} 
0 & \text{for } t \in (\pi, 2\pi] \\
n^{1-(\log n)^{-1/2}} & \text{for } t \in \left(\frac{2\pi}{n+1}, \frac{2\pi}{n}\right], \ n \in N \setminus \{1\}.
\end{cases}
\]

For fixed \( q > 1 \), we can choose \( n_0 \in N \) such that \( q - q(\log n)^{-1/2} \geq 1 \), for all \( n \geq n_0 \). Then we have

\[
\int_0^{2\pi} \psi(t) \, dt = \sum_{n=2}^{\infty} \frac{n^{q-(\log n)^{-1/2}}}{n(n+1)} \geq \sum_{n=n_0}^{\infty} \frac{1}{n+1} = \infty.
\]

Thus, for each \( q > 1 \) \( \psi^{q+} \neq L^1 \); so that by 2.2 (b), \( f \notin \cup_{q>1} N^q \).

On the other hand, since \( n^{-(\log n)^{-1/2}} = \exp \left( -\sqrt{\log n} \right) \), we have

\[
\int_0^{2\pi} \psi(t) \log^+ \psi(t) \, dt = \sum_{n=3}^{\infty} \frac{\log n}{(n+1) \exp(\sqrt{\log n})} \left( 1 - \frac{1}{\sqrt{\log n}} \right) < \sum_{n=3}^{\infty} \frac{6! \log n}{n \log^3 n} < \infty.
\]
Thus $\psi = \log|f^*|$ belongs to the Zygmund class $L\log L$. By Theorem 1.5 of [2] and 2.2 (c), we conclude that $f \in M$. So that $\bigcup_{q>1} N^q \neq M$. □

Proof of (ii). The for each $s > 0$ and $q > 1$ the inclusion $H^s \subseteq N^q$ follows from inequality $\log^{+q} x \leq (q/s)^q x^s$, $x \geq 0$. Hence, $\cup_{s>0} H^s \subseteq \cap_{q>1} N^q$. For the proof of $\cup_{s>0} H^s \neq \cap_{q>1} N^q$, let $g$ be the function defined by

$$g(z) = \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log \chi(t) \, dt \right),$$

where $\chi(t)$ is a step function defined by

$$\chi(t) = n^{\log n}, \quad \text{for all } t \in \left( \frac{2\pi}{n+1}, \frac{2\pi}{n} \right], \quad n \in \mathbb{N}.$$

For each $q \geq 1$, we have

$$\int_0^{2\pi} \log^q \chi(t) \frac{dt}{2\pi} = \sum_{n=1}^{\infty} \frac{\log^q n}{n(n+1)} < \infty,$$

since $\log^q n < \sqrt{n}$ for sufficiently large $n$. So by 2.2 (b), $g \in \cap_{q>1} N^q$.

On the other hand, for each $s > 0$ we have

$$\int_0^{2\pi} |\chi(t)|^s \frac{dt}{2\pi} = \sum_{n=1}^{\infty} \frac{n^{s\log n}}{n(n+1)} = \infty,$$

since $s \log n > 1$ for all $n > \exp(1/s)$. Hence, $\chi \notin L^s$ for any $s > 0$; so that, by 2.2 (d), $f \notin \cup_{p>0} H^p$. This completes the proof of (ii). □

Proof of (iii). The inclusion $N^q \subseteq N^p$, for $q > p > 1$, is obvious. Hence,

$$\bigcup_{q>p} N^q \subseteq N^p \subseteq \bigcap_{1<q<p} N^q.$$

Define

$$F_p(z) = \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \xi_p(t) \, dt \right),$$

where

$$\xi_p(t) = \begin{cases} 
0 & \text{for } t \in (\pi, 2\pi] \\
\frac{1}{n} \left( 1 - (\log n)^{-1/2} \right) & \text{for } t \in \left( \frac{2\pi}{n+1}, \frac{2\pi}{n} \right], \quad n \in \mathbb{N} \setminus \{1\}. 
\end{cases}$$
Proceeding as in the proof of (i) we obtain \( \xi_p \in L^p \setminus \bigcup_{q>p} L^q \), and this implies

\[
F_p \in N^p \setminus \bigcup_{q>p} N^q.
\]

For any \( p > 1 \), consider

\[
G_p(z) = \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \varphi_p(t) \, dt \right),
\]

where \( \varphi_p(t) = n^{1/p} \) for

\[
t \in \left( \frac{2\pi}{n + 1}, \frac{2\pi}{n} \right), \quad n \in \mathbb{N}.
\]

Then we have

\[
\frac{2\pi}{n+1} \int_0^{2\pi} |\varphi_p(t)|^q \, dt = \sum_{n=1}^{\infty} \frac{n^2}{n(n+1)}.
\]

Clearly \( \varphi_p \in \cap_{1 < q < p} L^q \setminus L^p \); so that, by 2.2 (b), \( G_p \in \cap_{1 < q < p} N^q \setminus N^p \).

This proves (iii), and so the proof of Theorem 3.1 is completed. \( \square \)

4. THE LIMIT RELATIONS IN THE CLASSES \( N^q \)

**Theorem 4.1.** Let \( q > 1 \) and let \( f \in N \). The following are equivalent statements.

(i) \( f \) belongs to the class \( N^q \).

(ii) \( \lim_{r \to 1} \frac{2\pi}{0} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi} = \frac{2\pi}{0} \log^+ |f^*(e^{i\theta})| \frac{d\theta}{2\pi} < \infty \).

(iii) \( f \) belongs to the class \( N^+ \) and \( \log^+ |f^*| \in L^q \).

(iv) \( \log^+ |f^*| \in L^q \) and

\[
\lim_{r \to 1} \frac{2\pi}{0} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi} = \frac{2\pi}{0} \log^+ |f^*(e^{i\theta})| \frac{d\theta}{2\pi}.
\]

(v) \( f \) belongs to the class \( M \) and \( \log^+ |f^*| \in L^q \).

**Proof.** (i) \( \iff \) (ii) Since \( f \in N \), by 2.1 the radial limit \( f^*(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta}) \) exists for almost every \( e^{i\theta} \). By Privalov’s theorem 2.3, \( f \) belongs to \( N^q \) if and only if \( \{ \log^+ |f(re^{i\theta})| \} \) \( (0 \leq r < 1) \) form a uniformly integrable family. This is (see [4], p. 13) equivalent to the limit relation (ii) with \( \log^+ |f^*| \in L^q \).

(iii) \( \iff \) (iv) follows in the same way as (i) \( \iff \) (ii), in view of the definition of the class \( N^+ \).
(i) ⇔ (iii) follows immediately from the canonical factorizations described in 2.2(a) and (b).

(v) ⇒ (iii) is obvious, since \( M \subset N^q \).

(i) ⇒ (v) From 2.2 (b) it follows that \( \log^+|f^*| \in L^q \). By Theorem 3.1 (i), \( N^q \subset M \); so that \( f \in M \). This completes the proof of Theorem. □

As a consequence of Theorem 4.1 we obtain the following result which is \( N^q \)-analogue of Smirnov’s theorem for the classes \( H^p \) (Koosis [3]).

**Corollary 4.2.** Let \( p \) and \( q \) be the numbers such that \( 1 < p < q \) and \( f \in N^p \). Then \( f \) belongs to the class \( N^q \) if and only if \( \log^+|f^*| \in L^q \).

**Proof.** Suppose \( \log^+|f^*| \in L^q \). Since \( f \in N^p \subset N^q \), by (iii) ⇒ (i) of Theorem 4.1, we conclude that \( f \in N^q \). Conversely, if \( f \in N^q \), then by (i) ⇒ (iv), it follows that \( \log^+|f^*| \in L^q \). □

**Remark 4.3.** We observe that the limit relations (ii) and (iv) of Theorem 4.1 are the analogues of Riesz’s theorem of ([3], p. 61) concerning the classes \( H^p \) \((0 < p < \infty)\).

5. Canonical factorization theorem

We give in this section, another proof of the Canonical factorization theorem 2.2 (b) for the classes \( N^q \) \((1 < q < \infty)\). As an immediate consequence of this proof, we obtain Privalov’s theorem 2.3. We will need the following result.

**Lemma 5.1.** Let \( q > 1 \) and let \( K \) be a bounded subset of \( L^q \). Then \( K \) forms an uniformly integrable family, i.e. for given \( \varepsilon > 0 \) there exists a \( \delta > 0 \) so that

\[
\int_E |f(\theta)| \frac{d\theta}{2\pi} < \delta \quad \text{for all} \quad f \in K,
\]

whenever \( E \subset T \) with its Lebesgue measure \( \mu(E) < \delta \).

**Proof.** This is an immediate consequence of (Gamelin [1], Ch. V, Theorem 1.1 and Corollary 1.2, p. 121). □

**Lemma 5.2.** \( N^q \subset N^+ \).

**Proof.** Let \( f \in N^q \). By the definition of \( N^q \), the family \( \log^+|f(re^{i\theta})| \) \((0 \leq r < 1)\) is bounded in \( L^q \); so by Lemma 5.1, it is uniformly integrable. Hence \( f \in N^+ \), i.e. \( N^q \subset N^+ \). □

**Theorem 5.3 (Canonical factorization theorem 2.2. (b)).** A function \( f \in N^q \) can be factored as follows

\[
f(z) = B(z)S(z)F(z),
\]
where $B(z)$ is the Blaschke product with respect to zeros of $f(z)$, $S(z)$ is a singular inner function and $F(z)$ is an outer function for the class $N^q$, i.e

\[
F(z) = \omega \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log |f^*(e^{it})| \, dt \right),
\]

where $|\omega| = 1$, $\log|f^*|$ and $\log^+|f^*|$ belong to $L^1$. Conversely, every such product $B(z)S(z)F(z)$ belongs to $N^q$.

Proof. Let $f \in N^q$. By Lemma 5.2, $f \in N^+$. In view of 2.2(a), $f$ can be expressed in the form (5.1), where $S(z)$ is given by (5.2), and $\log|f^*| \in L^1$. By Fatou’s lemma, we have

\[
\int_0^{2\pi} \log^+ |f^*(e^{i\theta})| \frac{d\theta}{2\pi} \leq \liminf_{r \to 1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi} < \infty,
\]

from which it follows that $\log^+|f^*| \in L^1$.

Conversely, suppose that $f(z)$ is given by (5.1) with $\log|f^*|$, $\log^+|f^*| \in L^1$. Let

\[
P(r, \theta - t) = \frac{1 - r^2}{1 - 2r \cos(\theta - r) + r^2}
\]

denote the Poisson kernel and let $\psi(t) = \log |f^*(e^{it})|$. Then from (5.2) it follows that

\[
\log^+ \left| F(re^{i\theta}) \right| = \left( \int_0^{2\pi} P(r, \theta - t)\psi(t) \frac{dt}{2\pi} \right)^+
\]

\[
\leq \int_0^{2\pi} P(r, \theta - t)\psi^+(t) \frac{dt}{2\pi}.
\]
Since $|f(z)| \leq |F(z)|$ for all $z \in D$, by (5.4) and Hölder’s inequality, for each $0 \leq r < 1$ we have

\[
\begin{align*}
\log^+ q |f(re^{i\theta})| &\leq \left( \int_0^{2\pi} P(r, \theta - t) \psi^+ (t) \frac{dt}{2\pi} \right)^q \\
&\leq \left( \int_0^{2\pi} P(r, \theta - t) \frac{dt}{2\pi} \right)^{q-1} \left( \int_0^{2\pi} P(r, \theta - t) \psi^+ (t) \frac{dt}{2\pi} \right)^q \\
&= \int_0^{2\pi} P(r, \theta - t) \log^+ q |f^*(e^{it})| \frac{dt}{2\pi}.
\end{align*}
\]

Using (5.5) and Fubini’s theorem

\[
\begin{align*}
\int_0^{2\pi} \log^+ q |f(re^{i\theta})| \frac{d\theta}{2\pi} &\leq \int_0^{2\pi} \log^+ q |f^*(e^{i\theta})| \frac{dt}{2\pi} \quad (0 \leq r < 1).
\end{align*}
\]

Thus, in view of $\log^+ q |f^*| \in L^1$, we conclude that $f \in N^q$. This completes the proof of Theorem.

**Corollary 5.4** (Theorem 2.3). A function $f$, holomorphic in $D$ belongs to the class $N^q$ if and only if the family $\log^+ q |f(re^{i\theta})| (0 \leq r < 1)$ is uniformly integrable.

**Proof.** Suppose that $f \in N^q$. From (5.6) and (5.3) we have

\[
\begin{align*}
\limsup_{r \to 1} \int_0^{2\pi} \log^+ q |f(re^{i\theta})| \frac{d\theta}{2\pi} &\leq \liminf_{r \to 1} \int_0^{2\pi} \log^+ q |f(re^{i\theta})| \frac{d\theta}{2\pi} < \infty.
\end{align*}
\]

Hence there exists

\[
\lim_{r \to 1} \int_0^{2\pi} \log^+ q |f(re^{i\theta})| \frac{d\theta}{2\pi} < \infty.
\]

So by (5.6) and (5.3), we obtain the limit relation

\[
\lim_{r \to 1} \int_0^{2\pi} \log^+ q |f(re^{i\theta})| \frac{d\theta}{2\pi} = \int_0^{2\pi} \log^+ q |f^*(e^{i\theta})| \frac{d\theta}{2\pi} < \infty.
\]

This means that the family $\log^+ q |f(re^{i\theta})| (0 \leq r < 1)$ is uniformly integrable. The converse is obvious. □
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