A CHARACTERIZATION OF AN SUBCLASS OF THE SMIRNOV CLASS

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Abstract. In this paper, we give a short proof of the Canonical factorization theorem for the Class \( N_+^* \) of holomorphic functions, introduced by Privalov with the notation \( C \) in [2]. We prove that the class \( N_+^* \) contains all polynomials, and hence it is a dense subset of the Smirnov class \( N^+ \).

1. Introduction

The Smirnov class \( N^+ \) consists of those functions \( f \) holomorphic on the unit disk \( D \) in the complex plane for which

\[
\lim_{r \to 1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi} = \int_0^{2\pi} \log^+ |f(e^{i\theta})| \frac{d\theta}{2\pi} < \infty.
\]

(The boundedness of the integrals on the left implies that \( f \) is in the Nevanlinna class \( N \), and so has non-tangential limits almost everywhere on the unit circle. It is this boundary function that we mean in the second integral).

As in [2, p. 89], where \( N_+^* \) is denoted as \( C \), a function \( f \in N \) is said to belong to the class \( N_+^* \) if there holds

\[
\lim_{r \to 1} \int_0^{2\pi} \log |f(re^{i\theta})| \frac{d\theta}{2\pi} = \int_0^{2\pi} \log |f(e^{i\theta})| \frac{d\theta}{2\pi} < \infty,
\]

which is equivalent to the fact that the family \( \{ \log |f(re^{i\theta})| : 0 \leq r < 1 \} \), is uniformly integrable if \( r \) is near to 1. This means (see [3,
that for given $\varepsilon > 0$, there exists $\delta > 0$ and $r_0 < 1$ so that

$$\int_E \left| \log |f(re^{i\theta})| \right| \frac{d\theta}{2\pi} < \varepsilon \quad (r_0 < r < 1),$$

whenever $E \subset [0, 2\pi]$ with its Lebesgue measure $|E| < \delta$.

By [1, p. 25], every function $f$ of class $N$ can be factored as

$$f(z) = B(z)\left(S_1(z)/S_2(z)\right)F(z),$$

where $B(z)$ is the Blaschke product with respect to zeros of $f(z)$, $S_k(z)$, $k = 1, 2$, are the singular inner functions with no common factor and $F(z)$ is an outer function for the class $N$, i.e.,

$$S_k(z) = \exp \left( -\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu_k(t) \right)$$

with positive singular measures $d\mu_k$, $k = 1, 2$, and

$$F(z) = \omega \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log |f(e^{it})| \, dt \right),$$

with $\omega$ a constant of unit modulus. Furthermore $\log |f(e^{it})| \in L^1(0, 2\pi)$ unless $f \equiv 0$. It is known that a function $f \in N$ belongs to $N^+$ if and only if $S_2 \equiv 0$. See, e.g., [1, p. 26] or [2, p. 89]. It was showed in Privalov [2, pp. 90-93] that a function $f \in N$ belongs to $N^+_s$ if and only if $S_1 \equiv 0$ and $S_2 \equiv 0$. This means that $f \in N^+_s$ cannot have a (nontrivial) singular factor.

In the next section, using a result of Stoll [4], we obtain a simple proof of this Canonical factorization. In Section 3, we observe that all univalent function is in class $N^+_s$. In Section 4, we show that the polynomials are dense in $N^+_s$, and therefore $N^+_s$ is a dense topological multiplicative submonoid of $N^+$.

2. A factorization theorem for the class $N^+_s$

**Theorem 2.1** ([2, Sec. 9.2, p. 93]) Every function $f \in N^+_s$ has a unique factorization of the form

$$f(z) = B(z)F(z),$$

where $B(z)$ is the Blaschke product with respect to zeros of $f(z)$, and $F(z)$ is an outer function. Conversely, every such product $B(z)F(z)$ belongs to $N^+_s$. 
For the proof of the Theorem, we will need three Lemmas. The following result was proved by Stoll [4, Theorem 4], for an arbitrary bounded symmetric domain $D$ in $\mathbb{C}^m$ with Bergman-Shilov boundary $B$ and $0 \in D$.

**Lemma 2.2.** If $F \in N^+$ is outer, then

\[
\lim_{r \to 1} \int_0^{2\pi} \left| \log |F(re^{i\theta})| - \log |F(e^{i\theta})| \right| \frac{d\theta}{2\pi} = 0.
\]

Conversely, if $F \in N^+$, $F(z) \neq 0$ for all $z \in D$, satisfies (2.1) then $F$ is outer.

The following lemma follows immediately from [1, p. 21, Lemma 1] and the definition of $N^+_s$.

**Lemma 2.3.** Every function $F \in N^+_s$ satisfies the condition (2.1) from Lemma 2.2.

The following lemma is proved in the proof of Theorem 2.10 of [1, p. 26], which is in fact the factorization theorem for elements of the Smirnov class $N^+$.

**Lemma 2.4** If $B(z)$ is an arbitrary Blaschke product, then

\[
\lim_{r \to 1} \int_0^{2\pi} \log |B(re^{i\theta})| \frac{d\theta}{2\pi} = 0.
\]

**Proof of Theorem 2.1.** Suppose first that $f \in N^+_s$. Since $N^+_s \subset N^+$, $f$ can be factored in the form $f = BSF$, where $B$, $S$, $F$ are as above. Put $G = SF$. By the inequality $|\log |xy|| \leq ||\log |x|| + ||\log |y||$, and the fact that $|B(z)| < 1$, we have

\[
\int_E \left| \log |G(re^{i\theta})| \right| \frac{d\theta}{2\pi} \leq \int_E \left| \log |f(re^{i\theta})| \right| \frac{d\theta}{2\pi} - \int_0^{2\pi} \log |B(re^{i\theta})| \frac{d\theta}{2\pi}
\]

for any measurable set $E \subset [0, 2\pi)$. From this and Lemma 2.4, we see that $\left\{ \left| \log |f(re^{i\theta})| \right| : r \to 1^- \right\}$ form a uniformly integrable family in the sense of (1.1). Hence $G$ is in $N^+_s$, and by Lemma 2.3, $G$ satisfies
(2.1). Since $G(z) \neq 0$ for all $z \in D$, by Lemma 2.2, we conclude that $G$ is outer. Therefore $f = BG$, as desired.

Conversely, assume that $f = BF$, where $B(z)$ is the Blaschke product with respect to zeros of $f(z)$, and $F(z)$ is an outer function. Then for any measurable set $E \subset [0, 2\pi)$, we have

$$
\int_E \log |f(re^{i\theta})| \frac{d\theta}{2\pi} \leq \int_E \log |F(re^{i\theta})| \frac{d\theta}{2\pi} - \int_E \log |B(re^{i\theta})| \frac{d\theta}{2\pi} 
\leq \int_0^{2\pi} \left[ \log |F(re^{i\theta})| - \log |F(e^{i\theta})| \right] \frac{d\theta}{2\pi} 
+ \int_E \log |F(e^{i\theta})| \frac{d\theta}{2\pi} - \int_E \log |B(re^{i\theta})| \frac{d\theta}{2\pi},
$$

whence by Lemmas 2.2 and 2.4, we conclude that $\left\{ |\log |G(re^{i\theta})|| : r \to 1^- \right\}$ form a uniformly integrable family (in the sense of (1.1)). Thus $f$ is in $N^+_\ast$, which completes the proof of Theorem. \qed

3. A CHARACTERIZATION OF UNIVALENT FUNCTIONS

A function holomorphic in a domain is said to be schlicht (or univalent) if it does not take any value twice; that is, if $f(z_1) \neq f(z_2)$ whenever $z_1 \neq z_2$. Let $H^p (0 < p \leq \infty)$ denote the classical Hardy space on the unit disk $D$. It is known (see [1, pp. 50–51]) that if $f$ is holomorphic and schlicht in $D$, then $f \in H^p$ for all $p < 1/2$, and its singular factor $S(z) \equiv 1$. As an immediate consequence of this fact and Theorem 2.1, we obtain the following result.

**Corollary. 3.1.** If $f$ is holomorphic and schlicht in $D$, then $f$ belongs to the class $N^+_\ast$. 

4. \( N_s^+ \) as a Dense Subclass of \( N^+ \)

The space \( N^+ \) with the metric \( \rho \) given by

\[
(4.1) \quad \rho(f, g) = \int_0^{2\pi} \log \left( 1 + \left| f(e^{i\theta}) - g(e^{i\theta}) \right| \right) \frac{d\theta}{2\pi}
\]

is an \( F \)-algebra, i.e., a topological vector space with a complete translation invariant metric in which multiplication is continuous (see [5] and [4]).

For any \( f \in N^+ \) put \( fr(z) = f(rz) \) \((0 < r < 1)\). Then by [5, Lemma 3], \( \rho(fr, f) \to 0 \) as \( r \to 1^- \). Since \( fr \) can be uniformly approximated by polynomials on the closed unit disk, it can be approximated in \( N^+ \) by polynomials. Hence, the polynomials are dense in \( N^+ \). By the inequality \( \left| \log |xy| \right| \leq \left| \log |x| \right| + \left| \log |y| \right| \), we see that \( N_s^+ \) is a multiplicative monoid. The following theorem shows that \( N_s^+ \) is separable, but is not complete with respect to the metric \( \rho \) given by (4.1).

**Theorem 4.1.** \( N_s^+ \) contains the set of all polynomials. Therefore, \( N_s^+ \) is a dense topological submonoid of \( N^+ \).

**Proof.** Since \( N_s^+ \) is a multiplicative monoid, it is sufficient to show that \( N_s^+ \) contains all polynomials of the form \( z - \alpha \) with a complex number \( \alpha \). It is easy to see that \( z - \alpha \) is in \( N_s^+ \) for all \( \alpha \) such that \( |\alpha| \neq 1 \). It is known (see [3, p. 85]) that the function \( \log (1/(1-z)) \) is in \( \bigcap_{0<p<\infty} H^p \), and hence \( \log (1/(e^{ic} - z)) \) is in \( H^1 \) for all real number \( c \). By the mean convergence theorem [1, p. 21] and the inequality \( \left| \log |\xi| \right| \leq \left| \log |\xi| \right| \), we conclude that the family \( \left\{ \left| \log (e^{ic} - re^{i\theta}) \right| : 0 \leq r < 1 \right\} \) is uniformly integrable. Therefore, \( z - e^{ic} \) is in \( N_s^+ \) for all real number \( c \). This completes the proof. \( \square \)

**References**

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